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A (p, q) VERSION OF BOURGAIN'S THEOREM

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ABSTRACT. Let $1 < p, q < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. We construct an orthonormal basis $\{b_n\}$ for $L^2(\mathbb{R})$ such that $\Delta_p(b_n)$ and $\Delta_q(\widehat{b_n})$ are both uniformly bounded in n. Here $\Delta_\lambda(f) \equiv \inf_{a\in\mathbb{R}} \left(\int |x-a|^\lambda |f(x)|^2 dx\right)^{\frac{1}{2}}$. This generalizes a theorem of Bourgain and is closely related to recent results on the Balian-Low theorem.

1. Introduction

Given a square integrable function $f \in L^2(\mathbb{R})$, we formally define the Fourier transform of f by

$$\widehat{f}(\gamma) = \int f(t) e^{-2\pi i t\gamma} dt,$$

where integration is over the real line $\mathbb{R}$. The uncertainty principle in harmonic analysis is the general statement that a function and its Fourier transform cannot both be "too well localized". For example, Heisenberg's inequality states that if $f \in L^2(\mathbb{R})$ is of norm one, then

$$\frac{1}{4\pi} \leq \Delta(f)\Delta(\widehat{f}). \tag{1.1}$$

Here $\Delta(\cdot)$ is defined by

$$\Delta(f) = \left(\int |t - \mu(f)|^2 |f(t)|^2 dt\right)^{\frac{1}{2}}, \tag{1.2}$$

where

$$\mu(f) = \int t|f(t)|^2 dt. \tag{1.3}$$

For an overview of recent mathematical work on the uncertainty principle we refer the reader to [FS], [B1], [HJ].

This paper deals with how the uncertainty principle constrains the time and frequency localization of the elements in an orthonormal basis for $L^2(\mathbb{R})$. We generalize a theorem of Bourgain on the construction of orthonormal bases which are uniformly well localized with respect to the (t^2, γ^2) weights implicit in (1.1). We consider the more general class of non-symmetric weights given by (t^p, γ^q), where $\frac{1}{p} + \frac{1}{q} = 1$.

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2. Background

2.1. **The (t^2, γ^2) weight.** The Balian-Low theorem [Bal], [Lo], [G2] is the classical example of an uncertainty principle for orthonormal bases. If, for a given $f \in L^2(\mathbb{R})$, we define the *Gabor system*, $\mathcal{G}(f, a, b) = \{f_{m,n} : m, n \in \mathbb{Z}\}$, by

$$f_{m,n}(t) = e^{2\pi itmb} f(t - na), \tag{2.1}$$

then the Balian-Low theorem states that when $\mathcal{G}(f, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ we have

$$\int |t|^2 |f(t)|^2 dt = \infty \quad \text{or} \quad \int |\gamma|^2 |\widehat{f}(\gamma)|^2 d\gamma = \infty. \tag{2.2}$$

In particular, either $\Delta(f_{m,n}) = \Delta(f) = \infty$ for all $m, n \in \mathbb{Z}$ or $\Delta(\widehat{f_{m,n}}) = \Delta(\widehat{f}) = \infty$ for all $m, n \in \mathbb{Z}$. Thus, if a Gabor system forms an orthonormal basis for $L^2(\mathbb{R})$, then its elements either have uniformly poor localization in time or uniformly poor localization in frequency. The Balian-Low theorem is true in much greater generality than above; e.g., [DJ], [GHHK], [BCM], [GH].

A recent result due to the authors together with W. Czaja and P. Gadziński shows that the (t^2, γ^2) weights used in (2.2) cannot be significantly weakened. In [BCGP] it was shown that there exists $f \in L^2(\mathbb{R})$ such that $\mathcal{G}(f, 1, 1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and such that, for each $d > 2$,

$$\int \frac{1 + |t|^2}{\log^d(|t| + 2)} |f(t)|^2 dt < \infty \tag{2.3}$$

and

$$\int \frac{1 + |\gamma|^2}{\log^d(|\gamma| + 2)} |\widehat{f}(\gamma)|^2 d\gamma < \infty. \tag{2.4}$$

In view of this result and the Balian-Low theorem, it is natural to ask what happens for general orthonormal bases, i.e., those which are not necessarily Gabor systems. Namely, can a general orthonormal basis have "uniform" localization with respect to the (t^2, γ^2) weights? This question was first posed by Balian [Bal] and answered by Bourgain [Bou] in 1986.

Bourgain showed that given any $\epsilon > 0$, there exists an orthonormal basis $\{b_n : n \in \mathbb{N}\}$ for $L^2(\mathbb{R})$ such that

$$\forall n \in \mathbb{N}, \quad \Delta(b_n) \leq \frac{1}{2\sqrt{\pi}} + \epsilon \quad \text{and} \quad \Delta(\widehat{b_n}) \leq \frac{1}{2\sqrt{\pi}} + \epsilon. \tag{2.5}$$

This basis is uniformly localized with respect to (t^2, γ^2) in the sense that the $\Delta(b_n)$ and $\Delta(\widehat{b_n})$ are uniformly bounded. To put this in perspective, note that there are $\psi \in \mathcal{S}(\mathbb{R})$, the Schwartz class, which generate wavelet orthonormal bases, $\{\psi_{m,n} : m, n \in \mathbb{Z}\}$, for $L^2(\mathbb{R})$ [LM]. Since $\psi \in \mathcal{S}(\mathbb{R})$, each $\Delta(\psi_{m,n})$ and $\Delta(\widehat{\psi_{m,n}})$ is finite. However, these variances are not uniformly bounded for any wavelet system, although their product may be; e.g., see [B1].

The constant $\frac{1}{2\sqrt{\pi}}$ in (2.5) is significant since $g(t) = 2^{1/4} e^{-\pi t^2}$ implies $\Delta(g) = \Delta(\widehat{g}) = \frac{1}{2\sqrt{\pi}}$. Moreover, it is well known that this choice of g gives equality in (1.1), i.e., the Gaussian is a minimizer for Heisenberg's inequality. Thus, each of the elements in Bourgain's basis is almost optimally localized with respect to Heisenberg's inequality.

2.2. **The (t^p, γ^q) weights.** The three results in the previous subsection give insight into the limits of the uncertainty principle for the t^2 and γ^2 weights. Our investigation in this paper deals with the more general weights t^p and γ^q, where $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p, q < \infty$. In this setting, one has the following analogue to the Balian-Low theorem which follows from work of Feichtinger and Gröchenig. Suppose $\epsilon > 0$, $f \in L^2(\mathbb{R})$, and $\mathcal{G}(f,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$. Then

$$\int |t|^{p+\epsilon}|g(t)|^2 dt = \infty \quad \text{or} \quad \int |\gamma|^{q+\epsilon}|\widehat{g}(\gamma)|^2 d\gamma = \infty. \tag{2.6}$$

This result is proven by combining Theorem 4.4 of [FG1] and Theorem 1 of [G1].

As in the case $(p,q) = (2,2)$, we proved that Gabor bases *are* possible if the t^p and γ^q weights are weakened slightly. In particular, it was shown in [BCGP] that there exists $f \in L^2(\mathbb{R})$ such that $\mathcal{G}(f,1,1)$ is an orthonormal basis for $L^2(\mathbb{R})$ and such that, for every $d > 2$,

$$\int \frac{1+|t|^p}{\log^d(|t|+2)}|f(t)|^2 dt < \infty \tag{2.7}$$

and

$$\int \frac{1+|\gamma|^q}{\log^d(|\gamma|+2)}|\widehat{f}(\gamma)|^2 d\gamma < \infty. \tag{2.8}$$

2.3. **Statement of the main result.** In view of the results in the previous subsection, we now consider the question of whether or not there is an analogue of Bourgain's theorem for the weights (t^p, γ^q). The following definition provides an appropriate generalization of $\Delta(\cdot)$.

Definition 2.1. Given $f \in L^2(\mathbb{R})$ and $\lambda > 0$, we define

$$\Delta_\lambda(f) = \inf_{a\in\mathbb{R}} \left(\int |t-a|^\lambda |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

It is easy to verify that when $\lambda = 2$ and $||f||_{L^2(\mathbb{R})} = 1$, this definition agrees with the one given by (1.2) and (1.3). Let $C_c^\infty(\mathbb{R})$ be the space of compactly supported, infinitely differentiable functions on $\mathbb{R}$. We now state our main result.

Theorem 2.2. *Assume $1 < p, q < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$. Fix $\epsilon > 0$ and $\varphi \in C_c^\infty(\mathbb{R})$ with $||\varphi||_{L^2(\mathbb{R})} = 1$. There exists an orthonormal basis, $\{b_n : n \in \mathbb{N}\} \subseteq C_c^\infty(\mathbb{R})$, for $L^2(\mathbb{R})$ such that*

$$\forall n \in \mathbb{N}, \quad \Delta_p(b_n) \le \left(\int |t|^p |\varphi(t)|^2 dt \right)^{\frac{1}{2}} + \epsilon \equiv C_{p,\varphi} + \epsilon \tag{2.9}$$

and

$$\forall n \in \mathbb{N}, \quad \Delta_q(\widehat{b_n}) \le \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma \right)^{\frac{1}{2}} + \epsilon \equiv C_{q,\varphi} + \epsilon. \tag{2.10}$$

For perspective, let us mention that Cowling and Price [CP] proved analogues of Heisenberg's inequality for the (t^p, γ^q) weights. Their results are quite general, but as a special case one has that if $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then there exists a constant $0 < K_{p,q}$ such that for all $f \in L^2(\mathbb{R})$ of norm one there holds

$$K_{p,q} \le \; [\Delta_p(f)]^{\frac{2}{p}} \; [\Delta_q(\widehat{f}\,)]^{\frac{2}{q}}. \tag{2.11}$$

Our main result, Theorem 2.2, allows one to construct orthonormal bases whose elements are almost optimally localized with respect to the Cowling-Price uncertainty principle, (2.11).

3. Preliminary lemmas

In this section we shall state several lemmas which will be needed to prove Theorem 2.2.

3.1. **Decay rates of inverses of matrices.** Theorem 3.2 relates the off-diagonal decay of an invertible matrix to the off-diagonal decay of its inverse. The results are due to Jaffard, [J], and have been further studied and simplified by Strohmer in [S]. We also note that Bourgain implicitly made use of similar implications in [Bou]. For example, see the transition between equations (2.11) and (2.12) in [Bou].

The following definition appears in [S].

Definition 3.1. Let $A = (A_{m,n})_{m,n \in \mathcal{I}}$ be a matrix, where the index set is $\mathcal{I} = \mathbb{Z}, \mathbb{N}$, or $\{0, \cdots, N-1\}$. Fix $s > 1$. We say that A belongs to $\mathcal{Q}_s$ if the coefficients $A_{m,n}$ satisfy

$$\exists C > 0 \text{ such that } \forall m, n \in \mathcal{I}, \quad |A_{m,n}| < \frac{C}{(1+|m-n|)^s}.$$

We say that A belongs to $\mathcal{E}_s$ if

$$\exists C > 0 \text{ such that } \forall m, n \in \mathcal{I}, \quad |A_{m,n}| < Ce^{-s|m-n|}.$$

Theorem 3.2 (Jaffard)**.** *Let $A : l^2(\mathcal{I}) \to l^2(\mathcal{I})$ be an invertible matrix, where $\mathcal{I} = \mathbb{Z}, \mathbb{N}$, or $\{0, \cdots, N-1\}$. Then*

$$A \in \mathcal{Q}_s \implies A^{-1} \in \mathcal{Q}_s$$

and

$$A \in \mathcal{E}_s \implies A^{-1} \in \mathcal{E}_{s'},$$

for some $0 < s' \leq s$.

The case $\mathcal{I} = \{0, 1, 2, \cdots, N-1\}$ should be interpreted as follows. We quote from [S]: "View the $n \times n$ matrix A_n as a finite section of an infinite dimensional matrix A. If we increase the dimension of A_n (and thus consequently the dimension of $(A_n)^{-1}$) we can find uniform constants independent of n such that the corresponding decay properties hold."

Let us next comment on the constants which arise in Jaffard's theorem. We restrict ourselves to the case $\mathcal{I} = \{0, 1, \cdots, N-1\}$. Suppose that the A_N are sections of the infinite matrix A, and that

$$\exists C > 0 \text{ such that } \forall N \geq 1 \text{ and } \forall\, j, k \in \mathcal{I}, \;\; |A_N(j,k)| \leq \frac{C}{1+|j-k|^s}.$$

Further, suppose for simplicity that there is a fixed $0 < r < 1$ such that

$$\forall N \geq 1, \; B_N \equiv I_N - A_N \text{ satisfies } ||B_N|| \leq r < 1,$$

where I_N is the $N \times N$ identity matrix. Jaffard's theorem then says that there exists C' such that

$$\forall N \geq 1 \text{ and } \forall\, j, k \in \mathcal{I}, \;\; |A_N^{-1}(j,k)| \leq \frac{C'}{1+|j-k|^s}.$$

The constant C' depends only on r, s, and C. The proof of this assertion can be obtained by examining Jaffard's proofs [J].

3.2. **A simple lemma.**

Lemma 3.3. *Fix $\varphi \in L^2(\mathbb{R})$ and $1 < q < \infty$. If $\int |\gamma|^q |\varphi(\gamma)|^2 d\gamma < \infty$, then*

$$\int |\gamma + N|^q |\varphi(\gamma)|^2 d\gamma \le 3^q |N|^q ||\varphi||_2^2 + \left(\frac{3}{2}\right)^q \int_{\mathbb{R}\setminus[-2N,2N]} |\gamma|^q |\varphi(\gamma)|^2 d\gamma \tag{3.1}$$

holds for all $N \ge 0$.

Proof. First note that

$$\int_{-2N}^{2N} |\gamma + N|^q |\varphi(\gamma)|^2 d\gamma \le 3^q |N|^q ||\varphi||_{L^2(\mathbb{R})}^2.$$

Next note that $2N \le \gamma$ implies $|\gamma + N|^q \le \left(\frac{3}{2}\right)^q |\gamma|^q$. Thus

$$\int_{2N}^{\infty} |\gamma + N|^q |\varphi(\gamma)|^2 d\gamma \le \left(\frac{3}{2}\right)^q \int_{2N}^{\infty} |\gamma|^q |\varphi(\gamma)|^2 d\gamma.$$

Likewise,

$$\int_{-\infty}^{-2N} |\gamma + N|^q |\varphi(\gamma)|^2 d\gamma \le \left(\frac{3}{2}\right)^q \int_{-\infty}^{-2N} |\gamma|^q |\varphi(\gamma)|^2 d\gamma.$$

This completes the proof. □

4. Finite, orthonormal, well localized systems

Lemma 4.1. *Assume $\frac{1}{p} + \frac{1}{q} = 1$, where $1 < p, q < \infty$. Fix $\epsilon > 0$ and $\varphi \in C_c^\infty(\mathbb{R})$ with $||\varphi||_{L^2(\mathbb{R})} = 1$. There is $K((p,q),\epsilon,\varphi)$ such that for each $K > K((p,q),\epsilon,\varphi)$, there exists an infinite orthonormal set $S_0 = S_0(K,\varphi) = \{s_n\}_{n=0}^\infty \subseteq C_c^\infty(\mathbb{R})$ satisfying*

$$\text{supp } s_n = \text{supp } \varphi \subseteq [-K/2, K/2], \tag{4.1}$$

$$\left(\int |t|^p |s_n(t)|^2 dt\right)^{\frac{1}{2}} \le \left(\int |t|^p |\varphi(t)|^2 dt\right)^{\frac{1}{2}} + \epsilon \equiv C_{p,\varphi} + \epsilon, \tag{4.2}$$

and

$$\left(\int |\gamma - nK|^q |\widehat{s_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \le \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \epsilon \equiv C_{q,\varphi} + \epsilon \tag{4.3}$$

for $n = 0, 1, 2, \cdots$.

Proof. Throughout the proof, C will denote various constants which are independent of K. C may depend on $(p,q), \varphi$, and N, all of which are fixed throughout the proof.

i. Let $\epsilon > 0$ and let $\varphi \in C_c^\infty(\mathbb{R})$ have $L^2(\mathbb{R})$ norm one. We may assume φ satisfies

$$|\widehat{\varphi}(\gamma)| \le \frac{C}{|\gamma|^N + 1}, \tag{4.4}$$

where $N > 4q, N \in \mathbb{N}$. Now define

$$\varphi_j(t) = e^{2\pi i j K t} \varphi(t), \quad j = 0, 1, 2, \cdots,$$

where K is a sufficiently large integer which will depend on $(p,q), \varphi, N$, and ϵ. We shall specify how large to take K during the proof. Next, define

$$h_0(t) = \varphi_0(t) \tag{4.5}$$

and

$$h_n(t) = \varphi_n(t) - \sum_{j=0}^{n-1} a_{n,j}\varphi_j(t), \quad n = 1, 2, \cdots, \tag{4.6}$$

where the $a_{n,j}$ are chosen to make h_n orthogonal to $\{\varphi_j\}_{j=0}^{n-1}$. This choice of $a_{n,j}$ implies that, for all $0 \leq l \leq n-1$,

$$\langle \varphi_n, \varphi_l \rangle = \sum_{j=0}^{n-1} a_{n,j} \langle \varphi_j, \varphi_l \rangle.$$

Rewriting this in matrix form, we have

$$Ga = g,$$

where $G = \begin{pmatrix} \langle \varphi_{n-1}, \varphi_{n-1}\rangle & \langle \varphi_{n-2}, \varphi_{n-1}\rangle & \cdots & \langle \varphi_0, \varphi_{n-1}\rangle \\ \langle \varphi_{n-1}, \varphi_{n-2}\rangle & \langle \varphi_{n-2}, \varphi_{n-2}\rangle & \cdots & \langle \varphi_0, \varphi_{n-2}\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle \varphi_{n-1}, \varphi_0\rangle & \langle \varphi_{n-2}, \varphi_0\rangle & \cdots & \langle \varphi_0, \varphi_0\rangle \end{pmatrix}$,

$$a = \begin{pmatrix} a_{n,n-1} \\ a_{n,n-2} \\ \cdots \\ a_{n,0} \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} \langle \varphi_n, \varphi_{n-1}\rangle \\ \langle \varphi_n, \varphi_{n-2}\rangle \\ \cdots \\ \langle \varphi_n, \varphi_0\rangle \end{pmatrix}.$$

Note that these matrices all depend on n, but we shall usually suppress this for economy of notation. When we wish to emphasize the dependence on n, we shall write $G = G_n$.

ii. First of all, observe that G is an invertible matrix since the finite set $\{\varphi_j\}_{j=0}^{n-1}$ is linearly independent by Proposition 1 of [HRT]. In particular, one also has that $\{a_{n,j}\}_{j=0}^{n-1}$ is uniquely determined.

To apply Jaffard's theorem, we also need to know that the spectrum of $G = G_n$ stays uniformly bounded away from 0 independent of n. Note that the matrix G is a Toeplitz matrix, and by (4.4) it has polynomial decay of order N off the main diagonal. In fact,

$$|G(j,k)| \leq \frac{C}{1 + K^N |j-k|^N} \leq \frac{C}{1 + |j-k|^N}. \tag{4.7}$$

For K large enough, the first inequality of (4.7) implies $G = G_n$ is diagonally dominant and has spectrum uniformly bounded away from 0.

iii. By Jaffard's theorem, G^{-1} has the same type of decay off its main diagonal as G, namely,

$$|G^{-1}(j,k)| \leq \frac{C}{1 + |j-k|^N}.$$

Also, the comments after the statement of Jaffard's theorem ensure that C is independent of K.

Therefore, noting that $a_{n,n-j}$ is the j-th element of the vector a,

$$\begin{aligned}|a_{n,n-j}| &\le \sum_{l=0}^{n-1} |G^{-1}(j,l)||g_l| = \sum_{l=0}^{n-1} |G^{-1}(j,l)||\langle \varphi_n, \varphi_{n-l-1}\rangle| \\ &\le \sum_{l=0}^{n-1} \left(\frac{C}{1+|j-l|^N}\right)\left(\frac{C}{1+K^N|l+1|^N}\right) \\ &\le \sum_{l=0}^{n-1} \frac{C}{1+|j-l|^N}\left(\frac{C}{K^N(l+1)^N}\right) \\ &\le \frac{C}{K^N}\sum_{l=0}^{n-1} \frac{1}{(1+|j-l|^N)}\frac{1}{|l+1|^N} \\ &\le \frac{C}{K^N}\sum_{l=1}^{\infty} \frac{1}{(1+|(j+1)-l|^N)}\frac{1}{|l|^N} \\ &\le \left(\frac{1}{K^N}\right)\frac{C}{|j+1|^N}.\end{aligned}$$

To see the last step, first note that

$$\sum_{1\le l\le \frac{j+1}{2}} \frac{1}{|l|^N(1+|j+1-l|^N)} \le \frac{1}{(1+|\frac{j+1}{2}|^N)}\sum_{l=1}^{\infty}\frac{1}{l^N}.$$

Combining this with a similar estimate for the remaining range of summation gives the desired inequality.

Thus, we have

$$|a_{n,j}| = |a_{n,n-(n-j)}| \le \frac{C}{K^N|n-j+1|^N}. \tag{4.8}$$

iv. Observe that

$$\sum_{j=0}^{n-1}|a_{n,j}| \le \frac{C}{K^N}\sum_{j=0}^{n-1}\frac{1}{|n-j+1|^N} \le \frac{C}{K^N}\sum_{j=2}^{n+1}\frac{1}{j^N} \le \frac{C}{K^N}. \tag{4.9}$$

Combining this and (4.6), we can estimate the localization of the $h_n(t)$.

$$\begin{aligned}\left(\int |t|^p|h_n(t)|^2dt\right)^{\frac12} &\le \left(\int |t|^p|\varphi_n(t)|^2dt\right)^{\frac12} + \sum_{j=0}^{n-1}|a_{n,j}|\left(\int |t|^p|\varphi_j(t)|^2dt\right)^{\frac12} \\ &= \left(\int |t|^p|\varphi(t)|^2dt\right)^{\frac12} + \left(\sum_{j=0}^{n-1}|a_{n,j}|\right)\left(\int |t|^p|\varphi(t)|^2dt\right)^{\frac12} \\ &\le \left(\int |t|^p|\varphi(t)|^2dt\right)^{\frac12} + \frac{C}{K^N}\left(\int |t|^p|\varphi(t)|^2dt\right)^{\frac12}.\end{aligned}$$

Thus for K large enough,

$$\int |t|^p|h_n(t)|^2dt \le C_{p,\varphi} + \frac{\epsilon}{2} \tag{4.10}$$

holds for all $n = 0, 1, 2, \cdots$.

v. We now estimate the localization of the $\widehat{h_n}(t)$:

$$\begin{aligned}
&\left(\int |\gamma - nK|^q |\widehat{h_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\leq \left(\int |\gamma - nK|^q |\widehat{\varphi}(\gamma - nK)|^2 d\gamma\right)^{\frac{1}{2}} + \left(\int |\gamma - nK|^q \left|\sum_{j=0}^{n-1} a_{n,j}\widehat{\varphi}(\gamma - jK)\right|^2 d\gamma\right)^{\frac{1}{2}} \\
&\leq \left(\int |\gamma|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \sum_{j=0}^{n-1} |a_{n,j}| \left(\int |\gamma - K(n-j)|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}}.
\end{aligned}$$

Using (4.8) and Lemma 3.3 we have

$$\begin{aligned}
&\sum_{j=0}^{n-1} |a_{n,j}| \left(\int |\gamma - K(n-j)|^q |\widehat{\varphi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\leq \sum_{j=0}^{n-1} |a_{n,j}| \left[|K(n-j)|^q ||\widehat{\varphi}||^2_{L^2(\mathbb{R})} + (3/2)^q \int |\gamma||\widehat{\varphi}(\gamma)|^2 d\gamma\right]^{\frac{1}{2}} \\
&\leq CK^{q/2} \sum_{j=0}^{n-1} |a_{n,j}||n-j|^{q/2} \leq CK^{q/2} \sum_{j=0}^{n-1} \frac{|n-j|^{q/2}}{K^N |n-j+1|^N} \\
&\leq \frac{C}{K^{N-q/2}}.
\end{aligned}$$

Thus, combining the above with K large enough gives

$$\left(\int |\gamma - nK|^q |\widehat{h_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq C_{q,\varphi} + \frac{\epsilon}{2} \tag{4.11}$$

for all $n = 0, 1, 2, \cdots$.

vi. It remains to normalize the h_n. First note that

$$||\varphi_n - h_n||_{L^2(\mathbb{R})} \leq \sum_{j=0}^{n-1} |a_{n,j}| \leq \frac{C}{K^N} \sum_{j=0}^{n-1} \frac{1}{|n-j+1|^N} \leq \frac{C}{K^N} \sum_{j=2}^{\infty} \frac{1}{j^N} = \frac{C}{K^N},$$

so that

$$1 = ||\varphi_n||_{L^2(\mathbb{R})} \leq ||h_n||_{L^2(\mathbb{R})} + ||h_n - \varphi_n||_{L^2(\mathbb{R})} \leq ||h_n||_{L^2(\mathbb{R})} + \frac{C}{K^N}$$

and

$$||h_n||_{L^2(\mathbb{R})} \leq ||\varphi||_{L^2(\mathbb{R})} + ||h_n - \varphi_n||_{L^2(\mathbb{R})} \leq 1 + \frac{C}{K^N}.$$

Thus we have

$$1 - \frac{C}{K^N} \leq ||h_n||_{L^2(\mathbb{R})} \leq 1 + \frac{C}{K^N}. \tag{4.12}$$

Finally, let $s_n(t) = h_n(t)/||h_n||_{L^2(\mathbb{R})}$. Taking K large enough and combining (4.10), (4.11), and (4.12) now shows that that (4.2) and (4.3) hold. □

Lemma 4.2. *Assume $\frac{1}{p}+\frac{1}{q}=1$, where $1<p,q<\infty$. Fix $\epsilon>0$ and $\varphi\in C_c^\infty(\mathbb{R})$ with $||\varphi||_{L^2(\mathbb{R})}=1$. Fix $K\in\mathbb{N}$ sufficiently large. For each $T>1$ there exists a finite orthonormal set,*

$$S=S(T,K)=S(T,K,\varphi)=\{s_{m,n}:0\le m<\lfloor T^{2/q}\rfloor,0\le n<\lfloor T^{2/p}\rfloor\}\subseteq C_c^\infty(\mathbb{R}),$$

of cardinality $\lfloor T^{2/p}\rfloor\lfloor T^{2/q}\rfloor\le T^2$ satisfying

$$\text{supp } s_{m,n}\subseteq\left[\frac{1}{2}T^{2/p},2T^{2/p}K\right], \tag{4.13}$$

$$\left(\int|t-Kn-T^{(2/p)}K|^p|s_{m,n}(t)|^2dt\right)^{\frac{1}{2}}\le C_{p,\varphi}+\epsilon, \tag{4.14}$$

and

$$\left(\int|\gamma-Km|^q|\widehat{s_{m,n}}(\gamma)|^2d\gamma\right)^{\frac{1}{2}}\le C_{q,\varphi}+\epsilon, \tag{4.15}$$

for all $0\le m<\lfloor T^{2/q}\rfloor,0\le n<\lfloor T^{2/p}\rfloor$. Here $C_{p,\varphi}$ and $C_{q,\varphi}$ are defined as in the previous lemma.

Proof. Let $\{s_m\}_{m=0}^{\lfloor T^{2/q}\rfloor-1}$ be defined using the system from the previous lemma. Define

$$s_{m,n}(t)=s_m(t-nK-T^{(2/p)}K)\quad\text{for}\quad 0\le m<T^{2/q}\quad\text{and}\quad 0\le n<T^{2/p}.$$

Now, (4.14) and (4.15) hold by the previous lemma. Since K was chosen large enough so that supp $\varphi\subseteq[-K/2,K/2]$, it follows that

$$\text{supp } s_{m,n}\subseteq\left[nK+T^{2/p}K-K/2,nK+T^{2/p}K+K/2\right],$$

so that all the $s_{m,n}$ are supported in

$$\left[T^{2/p}K-K/2,(T^{2/p}-1)K+T^{2/p}K+K/2\right]\subseteq\left[\frac{1}{2}T^{2/p},2KT^{2/p}\right].$$

□

Lemma 4.3. *Fix $\epsilon>0$ and $\varphi\in C_c^\infty(\mathbb{R})$. There exists a constant C such that for each $S(T,K)$ as in the previous lemma and for every $\Phi\in$ span $S(T,K)$ and each $0\le y\le T^{2/q}K$ one has*

$$\int|\gamma-y|^q|\widehat{\Phi}(\gamma)|^2d\gamma\le CK^qT^2||\Phi||^2_{L^2(\mathbb{R})}. \tag{4.16}$$

Proof. i. First note that

$$\int_{-2T^{2/q}K}^{2T^{2/q}K}|\gamma-y|^q|\widehat{\Phi}(\gamma)|^2d\gamma\le 3^qK^qT^2||\Phi||^2_{L^2(\mathbb{R})}.$$

ii. Recall that

$$\text{span } S(T,K)=\text{span}\{\varphi_{m,n}:0\le m<\lfloor T^{2/q}\rfloor,0\le n<\lfloor T^{2/p}\rfloor\},$$

where

$$\varphi_{m,n}(t)=e^{2\pi iKmt}\varphi(t-nK-T^{2/p}K).$$

Next, note that for K large enough, $\{\varphi_{m,n}:m,n\in\mathbb{Z}\}$ is a Riesz basis for its closed linear span. To see this, it suffices to show that $\{g_{m,n}:m,n\in\mathbb{Z}\}=\mathcal{G}(\varphi,K,K)$ is a Riesz basis for its closed linear span. Using Theorem 9 of Chapter

1, Section 8 in [Y] this is equivalent to proving that the Gram matrix $G_{(j,k),(l,m)} = \langle g_{j,k}, g_{l,m}\rangle$ defines a bounded positive operator on $l^2(\mathbb{Z}\times\mathbb{Z})$. Since $\varphi \in \mathcal{S}(\mathbb{R})$ one may directly verify that the Gram matrix is positive and bounded for all large enough K. In particular, for K large enough one can use Schur's test (see Lemma 6.2.1 in [G2]) to show that $M \equiv G - I$ satisfies $||M|| < \frac{1}{2}$, where I is the identity matrix, and $||\cdot||$ denotes the operator norm induced by the $l^2(\mathbb{Z}\times\mathbb{Z})$ norm. Hence there exists K_0 such that for all $K > K_0$, $\{\varphi_{m,n}\}$ is a Riesz basis for its closed linear span.

By the definition of Riesz basis there exist $0 < A \le B < \infty$ such that for each finite sum $\Phi(t) = \sum d_{m,n}\varphi_{m,n}(t)$,

$$A\sum |d_{m,n}|^2 \le ||\Phi||^2_{L^2(\mathbb{R})} \le B\sum |d_{m,n}|^2. \tag{4.17}$$

For us the constants A, B can be chosen independent of T and K. To see this, first note that the proof of Theorem 9 in Chapter 1, Section 8 of [Y] shows that one may take $B = ||G||$ and $A = ||G^{-1}||^{-1}$, where G is the Gram matrix defined above. A direct calculation with the Gram matrix shows that since $\varphi \in \mathcal{S}(\mathbb{R})$, the norm of the Gram matrix and its inverse approach 1 as $K \to \infty$.

We may conclude that there exists C, independent of T and K, such that for each $\Phi = \sum d_{m,n}\varphi_{m,n} \in \operatorname{span} S(T,K)$,

$$\left(\sum |d_{m,n}|^2\right)^{\frac{1}{2}} \le C||\Phi||_{L^2(\mathbb{R})}. \tag{4.18}$$

Here, and below, sums are over $0 \le m < \lfloor T^{2/q}\rfloor, 0 \le n < \lfloor T^{2/p}\rfloor$.

iii. We need to show that

$$\int_{2T^{2/q}K}^{\infty} |\gamma - y|^q |\widehat{\Phi}(\gamma)|^2 d\gamma \le CK^qT^2||\Phi||^2_{L^2(\mathbb{R})}. \tag{4.19}$$

iii.a. First note that since $0 \le y \le T^{2/q}K$, we have

$$\left(\int_{2T^{2/q}K}^{\infty} |\gamma - y|^q |\widehat{\Phi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \le \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q |\widehat{\Phi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}}. \tag{4.20}$$

iii.b. To estimate the right side of (4.20) we begin as follows:

$$\begin{aligned}\left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q |\widehat{\Phi}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} &= \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q \left|\sum_{m,n} d_{m,n}\widehat{\varphi_{m,n}}(\gamma)\right|^2 d\gamma\right)^{\frac{1}{2}}\\ &\le \sum_{m,n}|d_{m,n}| \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q \left|\widehat{\varphi_{m,n}}(\gamma)\right|^2 d\gamma\right)^{\frac{1}{2}}\\ &\le ||d_{m,n}||_{l^2(\mathbb{Z}^2)} \sum_{m,n} \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q \left|\widehat{\varphi_{m,n}}(\gamma)\right|^2 d\gamma\right)^{\frac{1}{2}}\\ &\le C||\Phi||_{L^2(\mathbb{R})} \sum_{m,n} \left(\int_{2T^{2/q}K}^{\infty} |\gamma|^q \left|\widehat{\varphi_{m,n}}(\gamma)\right|^2 d\gamma\right)^{\frac{1}{2}}. \qquad (4.21)\end{aligned}$$

Next note that

$$|\widehat{\varphi_{m,n}}(\gamma)| = |\widehat{\varphi}(\gamma - Km)|. \tag{4.22}$$

Thus,

$$\sum_{m,n}\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q\left|\widehat{\varphi_{m,n}}(\gamma)\right|^2 d\gamma\right)^{\frac{1}{2}}=\sum_{m,n}\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q\left|\widehat{\varphi}(\gamma-mK)\right|^2 d\gamma\right)^{\frac{1}{2}}$$

$$=\lfloor T^{2/p}\rfloor\sum_{m=0}^{\lfloor T^{2/q}\rfloor-1}\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q|\widehat{\varphi}(\gamma-mK)|^2d\gamma\right)^{\frac{1}{2}}$$

$$\leq T^2\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q|\widehat{\varphi}(\gamma-T^{2/q}K)|^2d\gamma\right)^{\frac{1}{2}}=T^2\left(\int_{T^{2/q}K}^{\infty}|\gamma+T^{2/q}K|^q|\widehat{\varphi}(\gamma)|^2d\gamma\right)^{\frac{1}{2}}$$

$$\leq T^2 2^{q/2}\left(\int_{T^{2/q}K}^{\infty}|\gamma|^q|\widehat{\varphi}(\gamma)|^2d\gamma\right)^{\frac{1}{2}}\leq\frac{CT^2}{|T^{2/q}K|^{2q}}.$$

The final inequality holds since $\varphi\in\mathcal{S}(\mathbb{R})$. Together with (4.21), this gives

$$\left(\int_{2T^{2/q}K}^{\infty}|\gamma|^q|\widehat{\Phi}(\gamma)|^2d\gamma\right)^{\frac{1}{2}}\leq\frac{C||\Phi||_{L^2(\mathbb{R})}}{K^{2q}T^2}. \tag{4.23}$$

Combining (4.20) and (4.23) yields (4.19), as desired.

iv. It remains to show that

$$\int_{-\infty}^{-2T^{2/q}K}|\gamma-y|^q|\widehat{\Phi}(\gamma)|^2d\gamma\leq CK^qT^2||\Phi||^2_{L^2(\mathbb{R})}.$$

This follows by calculations similar to those in part *iii.* The proof is complete. □

5. A (p, q) version of Bourgain's theorem

We are now ready to prove our main result, Theorem 2.2. The proof follows that of Bourgain [Bou], which, in turn, depends on an idea of W. Rudin [R] used to construct certain orthonormal bases for $H^2(B)$, where B is the unit ball of $\mathbb{C}^n$.

Proof of Theorem 2.2. Throughout the proof C will denote various constants which are independent of n, T_n, K, Θ, and any indices.

Let $\{f_n\}_{n\in\mathbb{N}}\subseteq C_c^\infty(\mathbb{R})$ be a sequence which is dense in the unit sphere of $L^2(\mathbb{R})$. Let $\epsilon>0$ and $\varphi\in C_c^\infty(\mathbb{R})$ be given. Let K be sufficiently large to ensure we may use Lemma 4.2 applied to $\frac{\epsilon}{2}$. The orthonormal basis we construct will be of the form $\bigcup_{n=1}^\infty B_n$, where B_n is a finite orthonormal subset of $C_c^\infty(\mathbb{R})$. We shall construct the B_n inductively.

i. Suppose $B_1,\cdots,B_{n-1}$ are already defined such that B_j is a finite orthonormal subset of $C_c^\infty(\mathbb{R})$ and the elements of $\bigcup_{j=1}^{n-1}B_j$ are mutually orthonormal. Define $F_n=f_n-P_{[B_1,\cdots,B_{n-1}]}f_n$, where $P_{[B_1,\cdots,B_{n-1}]}$ is the orthogonal projection of $L^2(\mathbb{R})$ onto

$$[B_1,\cdots,B_{n-1}]\equiv\operatorname{span}\bigcup_{l=1}^{n-1}B_l.$$

For the base case of the induction we simply let $F_1=f_1$. Using F_n, we now prepare to construct B_n.

i.a. Note that

$$||F_n||^2_{L^2(\mathbb{R})}\leq 1 \tag{5.1}$$

because $F_n\perp P_{[B_1,\cdots,B_n-1]}f_n$ and $||f_n||_{L^2(\mathbb{R})}=1$.

i.b. Since f_n and all elements of the B_j are in $C_c^\infty(\mathbb{R})$, it follows that $F_n \in C_c^\infty(\mathbb{R})$.

Choose $T_n > 2$ large enough so that

$$\frac{\lfloor T_n^{2/p}\rfloor \lfloor T_n^{2/q}\rfloor}{T_n^2} > \frac{1}{4}, \tag{5.2}$$

$$\text{supp } F_n \subseteq \left[-\frac{1}{2}T_n^{2/p}, \frac{1}{2}T_n^{2/p}\right], \tag{5.3}$$

$$\text{supp } b \subseteq [-\frac{1}{2}T_n^{2/p}, \frac{1}{2}T_n^{2/p}] \quad \text{for all} \quad b \in \bigcup_{j=1}^{n-1} B_j, \tag{5.4}$$

and

$$\forall y \geq 1, \quad \int_{\mathbb{R}\setminus[-2y,2y]} |\gamma|^q |\widehat{F_n}(\gamma)|^2 d\gamma \leq \frac{T_n}{1+|y|^N}, \tag{5.5}$$

where we have used the fact that $\widehat{F_n} \in \mathcal{S}(\mathbb{R})$ ($F_n \in C_c^\infty(\mathbb{R})$).

ii. Let

$$S = S(T_n, K) = \{s_{j,k}^n : 0 \leq j < \lfloor T_n^{(2/p)} \rfloor \text{ and } 0 \leq k < \lfloor T_n^{(2/q)} \rfloor\}$$

be the system from Lemma 4.2 applied with $\frac{\epsilon}{2}$ instead of ϵ. We shall switch from the double indexing (j,k) to single indexing, and enumerate the elements of the system as $\{s_l^n\}_{l=1}^{\lfloor T_n^{2/p}\rfloor \lfloor T_n^{2/q}\rfloor}$. If l_1, l_2 are the indices for which $s_l^n = s_{l_1,l_2}^n$, let

$$x(s_l^n) = Kl_1 + T_n^{(2/p)}K \quad \text{and} \quad y(\widehat{s_l^n}) = Kl_2,$$

so that by Lemma 4.2

$$\left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt\right)^{\frac{1}{2}} \leq C_{p,\varphi} + \frac{\epsilon}{2} \tag{5.6}$$

and

$$\left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq C_{q,\varphi} + \frac{\epsilon}{2}. \tag{5.7}$$

Note that

$$T_n^{(2/p)}K \leq x(s_j^n) \leq 2KT_n^{2/p} \quad \text{and} \quad 0 \leq y(\widehat{s_j^n}) \leq KT_n^{2/q}. \tag{5.8}$$

Let $0 < \Theta < \frac{1}{4}$ be sufficiently small and be fixed throughout the proof. We shall be more precise later about how small to take Θ. For now, note that K is fixed throughout the proof, so that Θ may depend on K (but not T_n). Let $\nu_n = \lfloor T_n^{2/p}\rfloor \lfloor T_n^{2/q}\rfloor$. Now define

$$\begin{aligned}
b_1^n(t) &= \frac{\Theta}{T_n}F_n(t) + \alpha_{n,1}s_1^n(t)\\
b_2^n(t) &= \frac{\Theta}{T_n}F_n(t) + \sigma_{n,1}s_1^n(t) + \alpha_{n,2}s_2^n(t)\\
&\ \ \vdots\\
b_{\nu_n}^n(t) &= \frac{\Theta}{T_n}F_n(t) + \sigma_{n,1}s_1^n(t) + \cdots + \sigma_{n,\nu_n-1}s_{\nu_n-1}^n(t) + \alpha_{n,\nu_n}s_{\nu_n}^n(t),
\end{aligned}$$

where the $\sigma_{n,j}$ and $\alpha_{n,j}$ are chosen to ensure that $\{b_j^n\}_{j=1}^{\nu_n}$ is orthonormal.

ii.a. The choice of $\sigma_{n,j}$ and $\alpha_{n,j}$ implies that

$$|1-\alpha_{n,j}| \le \frac{\Theta}{T_n} \quad \text{for} \quad j=1,2,\cdots,T_n^2 \tag{5.9}$$

and

$$|\sigma_{n,j}| \le \frac{\Theta}{T_n^2} \quad \text{for} \quad j=1,2,\cdots,T_n^2-1. \tag{5.10}$$

To see this, first note that $\{F_n\}\cup S(T_n,K)$ is an orthogonal set. Therefore, $\{b_j^n\}_{j=1}^{\nu_n}$ being orthonormal implies that for $l=1,2,\cdots,T_n^2$ we have

$$0=\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}+\sigma_{n,1}^2+\cdots+\sigma_{n,l-1}^2+\sigma_{n,l}\alpha_{n,l} \tag{5.11}$$

and for $l=1,2,\cdots,T_n^2-1$

$$\alpha_{n,l}^2=1-\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}-\sigma_{n,1}^2-\cdots-\sigma_{n,l-1}^2. \tag{5.12}$$

ii.b. Using (5.11) and (5.12) we shall now prove (5.9) and (5.10) by induction. The case $j=1$ of (5.9) holds since (5.12) implies

$$1=\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}+\alpha_{n,1}^2.$$

Since $2<T_n$ and $\Theta<\frac{1}{4}$, we may choose $0<\alpha_{n,1}\le 1$. Therefore,

$$|1-\alpha_{n,1}|\le|1-\alpha_{n,1}^2|\le\frac{\Theta^2}{T_n^2}\le\frac{\Theta}{T_n}.$$

Using this, the case $j=1$ of (5.10) now follows since, by (5.11),

$$0=\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}+\alpha_{n,1}\sigma_{n,1},$$

which implies

$$|\sigma_{n,1}|\le\frac{\Theta^2}{T_n^2}\frac{1}{|\alpha_{n,1}|}\le\frac{\Theta^2}{T_n^2}\frac{1}{(1-\Theta/T_n)}\le\frac{\Theta}{T_n^2}.$$

The last inequality holds because $\Theta<\frac{1}{4}$ and $T_n>2$.

ii.c. Next, assume $|\sigma_{n,j}|\le\frac{\Theta}{T_n^2}$ holds for $j<l$. We may once again choose $0<\alpha_{n,l}\le 1$. Since the cardinality of $S(T_n,K)$ is at most T_n^2,

$$|1-\alpha_{n,l}|\le|1-\alpha_{n,l}^2|\le\frac{\Theta^2}{T_n^2}+\sum_{j=1}^{l-1}\sigma_{n,j}^2\le\frac{\Theta^2}{T_n^2}+T_n^2\frac{\Theta^2}{T_n^4}\le 2\frac{\Theta^2}{T_n^2}\le\frac{\Theta}{T_n},$$

and (5.9) follows by induction. For (5.10), assume that $|\sigma_{n,j}|\le\frac{\Theta}{T_n^2}$ for $j<l$ and $|1-\alpha_{n,l}|\le\frac{\Theta}{T_n}$. Thus,

$$|\sigma_{n,l}|\le\frac{1}{|\alpha_{n,l}|}\left(\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})}+\sum_{j=1}^{l-1}\sigma_{n,j}^2\right)\le\frac{1}{(1-\Theta/T_n)}\left(2\frac{\Theta^2}{T_n^2}\right)\le\frac{\Theta}{T_n^2},$$

and (5.10) holds by induction.

iii. By (5.9) and (5.10), we know that $\sigma_{n,j}$ is close to zero and $\alpha_{n,j}$ is close to one. Thus, we expect to have b_j^n close to s_j^n. In fact,

$$||b_j^n - s_j^n||_{L^2(\mathbb{R})} \leq 3\frac{\Theta}{T_n}. \tag{5.13}$$

To see this, note that by (5.9) and (5.10)

$$\begin{aligned}
||b_j^n - s_j^n||_{L^2(\mathbb{R})} &\leq ||b_j^n - \alpha_{n,j}s_j^n||_{L^2(\mathbb{R})} + |1-\alpha_{n,j}| \\
&\leq ||b_j^n - \alpha_{n,j}s_j^n||_{L^2(\mathbb{R})} + \frac{\Theta}{T_n} \\
&= \left(\frac{\Theta^2}{T_n^2}||F_n||^2_{L^2(\mathbb{R})} + \sum_{k=1}^{j-1}|\sigma_{n,k}|^2\right)^{\frac{1}{2}} + \frac{\Theta}{T_n} \\
&\leq \left(\frac{\Theta^2}{T_n^2} + \left(T_n^2\frac{\Theta^2}{T_n^4}\right)\right)^{\frac{1}{2}} + \frac{\Theta}{T_n} \leq 3\frac{\Theta}{T_n}.
\end{aligned}$$

iv. Let us now prove that

$$\Delta_p(b_j^n) \leq \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt\right)^{\frac{1}{2}} + CK^{p/2}\Theta \leq C_{p,\varphi} + \epsilon. \tag{5.14}$$

Using (5.8), (5.13), and the fact that the b_j^n are supported in $[-\frac{1}{2}T_n^{2/p}, 2T_n^{2/p}K]$ (since F_n and $s_{n,j}$ are), we have

$$\begin{aligned}
&\left(\int |t - x(s_j^n)|^p |b_j^n(t)|^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\int |t - x(s_j^n)|^p |b_j^n(t) - s_j^n(t)|^2 dt\right)^{\frac{1}{2}} + \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt\right)^{\frac{1}{2}} \\
&\leq |2T_n^{2/p}K + 2KT_n^{2/p}|^{p/2}||b_j^n - s_j^n||_{L^2(\mathbb{R})} + \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt\right)^{\frac{1}{2}} \\
&\leq CK^{p/2}T_n||b_j^n - s_j^n||_{L^2(\mathbb{R})} + \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt\right)^{\frac{1}{2}} \\
&\leq CK^{p/2}\Theta + \left(\int |t - x(s_j^n)|^p |s_j^n(t)|^2 dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Assume Θ was chosen small enough to ensure $C\Theta K^{p/2} < \frac{\epsilon}{2}$. Thus, by (5.6) we have

$$\Delta_p(b_j^n) \leq C_{p,\varphi} + CK^{(p/2)}\Theta < C_{p,\varphi} + \epsilon. \tag{5.15}$$

v. Here we shall prove that

$$\Delta_q(\widehat{b_j^n}) \leq \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + C\Theta K^{(q/2)} < C_{q,\varphi} + \epsilon.$$

v.a. First we show that

$$\left(\int |\gamma - y(\widehat{s_j^n})|^q |\frac{\Theta}{T_n}\widehat{F_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \leq C\Theta K^{(q/2)}. \tag{5.16}$$

This follows from (5.5), (5.8), and Lemma 3.3:

$$\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{F_n}(\gamma)|^2 d\gamma \le 3^q |y(\widehat{s_j^n})|^q ||\widehat{F_n}||^2_{L^2(\mathbb{R})} + \frac{(3/2)^q T_n}{1 + |y(\widehat{s_j^n})|^M}$$
$$\le CT_n^2 K^q + CT_n \le CT_n^2 K^q.$$

v.b. Next, we show that

$$\left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{b_j^n}(\gamma) - \widehat{s_j^n}(\gamma) - \frac{\Theta}{T_n}\widehat{F_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \le C\Theta K^{(q/2)}.$$

Let $\Psi(\gamma) = \widehat{b_j^n}(\gamma) - \widehat{s_j^n}(\gamma) - \frac{\Theta}{T_n}\widehat{F_n}(\gamma)$. Note that Ψ is in the span of $S(T_n, K)$. Thus, using (5.8), (5.10), and Lemma 4.3,

$$\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{\Psi}(\gamma)|^2 d\gamma \le CT_n^2 K^q ||\Psi||^2_{L^2(\mathbb{R})} = CT_n^2 K^q \sum_{l=1}^{\lfloor T_n^{2/p}\rfloor \lfloor T_n^{2/q}\rfloor - 1} |\sigma_{n,l}|^2$$
$$\le CT_n^2 K^q \left(T_n^2 \frac{\Theta^2}{T_n^4}\right) = CK^q\Theta^2.$$

v.c. Combining the estimates from *v.a* and *v.b* we have

$$\begin{aligned}
\Delta_q(\widehat{b_j^n}) &\le \left(\int |\gamma - y(\widehat{s_j^n})|^q |b_j^n(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\le \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&+ \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{b_j^n}(\gamma) - \widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\le \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \left(\int |\gamma - y(\widehat{s_j^n})|^q |\frac{\Theta}{T_n}\widehat{F_n}(\gamma)|^2 \gamma\right)^{\frac{1}{2}} \\
&+ \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{b_j^n}(\gamma) - \widehat{s_j^n}(\gamma) - \frac{\Theta}{T_n}\widehat{F_n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} \\
&\le \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + \Theta CK^{(q/2)} + \Theta CK^{(q/2)} \\
&= \left(\int |\gamma - y(\widehat{s_j^n})|^q |\widehat{s_j^n}(\gamma)|^2 d\gamma\right)^{\frac{1}{2}} + C\Theta K^{(q/2)}.
\end{aligned}$$

Assume Θ was chosen small enough so that $C\Theta K^{q/2} < \frac{\epsilon}{2}$. Thus,

$$\Delta_q(\widehat{b_j^n}) \le C_{q,\varphi} + CK^{(q/2)}\Theta < C_{q,\varphi} + \epsilon. \tag{5.17}$$

vi. Having shown that all the elements of $B = \bigcup_{j=1}^{\infty} b_j^n$ have the desired localization, it only remains to prove that B is complete. To see this, note that, by (5.2) and the definition of F_n, we have

$$\begin{aligned}||P_{[B_1,\cdots,B_k]}f_k||^2_{L^2(\mathbb{R})} &= ||P_{[B_1,\cdots,B_{k-1}]}f_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}f_k||^2_{L^2(\mathbb{R})} \\ &= ||P_{[B_1,\cdots,B_{k-1}]}f_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}(F_k + P_{[B_1,\cdots,B_{k-1}]}f_k)||^2_{L^2(\mathbb{R})} \\ &= 1 - ||F_k||^2_{L^2(\mathbb{R})} + ||P_{[B_k]}F_k||^2_{L^2(\mathbb{R})} \\ &= 1 - ||F_k||^2_{L^2(\mathbb{R})} + \sum_{j=1}^{\lfloor T_k^{2/p}\rfloor\lfloor T_k^{2/q}\rfloor} |\langle F_k, b_j^k\rangle|^2 \\ &= 1 - ||F_k||^2_{L^2(\mathbb{R})} + \lfloor T_k^{2/p}\rfloor\lfloor T_k^{2/q}\rfloor \left(\frac{\Theta}{T_k}||F_k||^2_{L^2(\mathbb{R})}\right)^2 \\ &\geq 1 - ||F_k||^2_{L^2(\mathbb{R})} + (\Theta/2)^2||F_k||^4_{L^2(\mathbb{R})} \\ &\geq (\Theta/2)^2.\end{aligned}$$

To see the final inequality, let $h(t) = 1 - t^2 + a^2t^4$ be defined on $[0,1]$, where $0 < a < \frac{1}{4}$ is fixed. It is easy to see that $h(t) \geq a^2$. Since $||F_n||_{L^2(\mathbb{R})} \leq 1$ and $\Theta < \frac{1}{4}$, the last step follows.

Now, suppose $y \in L^2(\mathbb{R})$ satisfies $\langle y, b\rangle = 0$ for all $b \in B$. If y is not identically zero, then $\tilde{y} = y/||y||_{L^2(\mathbb{R})}$ is in the unit sphere of $L^2(\mathbb{R})$ and there exists f_{n_k} such that $f_{n_k} \to \tilde{y}$ in $L^2(\mathbb{R})$ as $k \to \infty$. Thus,

$$0 < \frac{\Theta}{2} \leq ||P_{[B_1,\cdots,B_{n_k}]}f_{n_k}||_{L^2(\mathbb{R})} \leq ||P_{[B]}f_{n_k}||_{L^2(\mathbb{R})} \to ||P_{[B]}\tilde{y}||_{L^2(\mathbb{R})} = 0,$$

where the limit is taken as $k \to \infty$. This contradiction shows that the orthonormal set B is complete, and hence it is an orthonormal basis for $L^2(\mathbb{R})$. □

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